International Journal of Theoretical Physics, Vol. 30, No. 9, 1991

Quantum Dynamics of the $SU(2)$-Skyrme Model in Nelson Stochastic Mechanics

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Received November 9, 1990

In the framework of Nelson stochastic mechanics the Skyrme $SU(2)$ model is quantized. A new term is added to a classical skyrmion mass. It coincides with the term obtained by Fujii *et al.* by modifying the canonical quantization. This example illustrates that stochastic mechanics as an alternative method of quantization is convenient for theories with collective coordinates and for nonlinear theories, as some problems related to operator ordering and modification of canonical formalism are naturally solved.

Following Fujii *et al.* (1986, 1987), we start from the Lagrangian of the $SU(2)$-invariant Skyrme model:

$$L(U_{L\rho}; \bar{x}, t) = \frac{f_\pi^2}{4} \operatorname{Tr}(U_{L\rho}, U_{L\rho}) + \frac{1}{32e_s^2} \operatorname{Tr}([U_{L\rho}, U_{L\rho}]^2) + \frac{m^2 f_\pi^2}{4} \operatorname{Tr}(U-1) + \text{h.c.} \tag{1}$$

Here $U_{L\rho} = \partial_\rho U U^+$ and $f_\pi = 93$ MeV. Collective coordinates $A(q(t))$, where $q^a(t)$ are real parameters ($a = 1, 2, 3$), are introduced in the same manner as in Adkins *et al.* (1983): $U = A(t)\sigma(\bar{x})A(t)^+$. Existence of a soliton solution with relevant boundary conditions (Fujii *et al.*, 1986, 1987) is also assumed: $\sigma(\bar{x}) = \exp[iF(r)\bar{\tau}\bar{n}]$, $r = |\bar{x}|$, $\bar{n} = \bar{x}/r$, $\bar{\tau} =$ Pauli matrix.

Within the standard approach (Adkins *et al.*, 1983), canonical quantization of the model considered is effected once the Lagrangian has been explicitly expressed in terms of the classically treated collective coordinates. Unlike this procedure of quantization, Fujii *et al.* (1986, 1987) proposed to

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0020-7748/91/0900-1229$06.50/0

treat collective coordinates $A(t)$ in the Lagrangian (1) from the beginning as quantum mechanical operators following a method of quantization of nonlinear theories that has been examined by various authors (Lin *et al.*, 1970; Sugano, 1971; Kimura, 1971; Kimura and Sugano, 1972). Fujii *et al.* (1986, 1987) found that in doing this a new additional mass term appears. This new mass term plays a role in stabilizing the rotating chiral soliton.

Unlike the standard canonical quantization method, Fujii *et al.* (1986, 1987), Lin *et al.* (1970), Sugano *et al.* (1971), Kimura (1971), and Kimura and Sagano (1972) take as starting point coordinates and velocities as quantum mechanical operators in the initial Lagrangian (1). Commutators of q^a and $\dot{q}^a$ are chosen so as to keep canonical commutation relations between q^a and canonical momentum p^a.

The quantum description in Nelson stochastic quantum mechanics (Dohrn and Guerra, 1978; Morato, 1982; Nelson, 1966; Blanchard *et al.*, 1987) (as an alternative formulation of quantum mechanics) is also provided within the Lagrangian formalism, without resorting to the Hamiltonian. It is interesting to compare the results of the operator approach by Fujii *et al.* (1986, 1987) and the results of stochastic quantum mechanics applied to the skyrmion quantization. Besides, it is important that so far the results of stochastic mechanics, applied to simple systems, coincided with those of canonical quantization. But here we see that to quantize the Skyrme model the formalism of stochastic mechanics appears to be more suitable and natural.

In the present paper we study the quantum dynamics of the rotational skyrmion within standard stochastic quantum mechanics to show that the results of such considerations agree with those of a *modified* canonical quantization carried out by Fujii *et al.* (1986, 1987; also see Lin *et al.*, 1970; Sugano, 1971; Kimura, 1971; Kimura and Sugano, 1972). Note that the problem of operator ordering, which makes handling these theories difficult both for operator quantization and path integration methods, is taken into account in the geometric structure of stochastic quantum mechanics itself.

The dynamics of a rotational skyrmion is analogous to the dynamics of a particle in a curved space. In stochastic mechanics dynamical variables $q^a(t)$ are rendered as random functions of time, satisfying stochastic equations describing the Brownian motion of a particle in the curved space with diffusion coefficient $\nu = h/2m$ (h = Planck's constant, m = mass of a particle).

The classical Lagrangian for the particle in the curved space, having the form

$$L = \frac{m}{2} g_{ik}(q)\dot{q}^i\dot{q}^k - V(q) \tag{2}$$

is replaced in stochastic mechanics by the Lagrangian (Dohrn and Guerra, 1978; Morato, 1982; Nelson, 1966; Blanchard *et al.*, 1987)

$$L = E\left[\frac{m}{2} g_{ik}(q) D_+ q^i D_- q^k\right] - V(q) \tag{3}$$

where time derivatives $\dot{q}^i$ are replaced by stochastic time derivatives:

$$D_\pm q^i = \lim_{\Delta t \to 0} E\left[\frac{q^i(t \pm \Delta t) - q^i(t)}{\Delta t}\middle| q^i(t)\right] \equiv \hat{b}^i_\pm \tag{4}$$

Here $E[\ldots]$ in (3) denotes averaging over the density of probability $\rho(q, t)$, and $E[\ldots | q^i]$ in (4) denotes averaging over conditional probabilities. For a suitable scalar function $F(q, t)$ one obtains (Dohrn and Guerra, 1978; Morato, 1982; Nelson, 1986; Blanchard *et al.*, 1987)

$$D_\pm F(q, t) = \frac{\partial F}{\partial t} + \hat{b}^j_\pm\ \partial_i F \pm \nu_{\mathrm{LB}} F \tag{5}$$

where $\Delta_{\mathrm{LB}} = (\sqrt{g})^{-1}\ \partial_j(\sqrt{g}\, g^{ij}\ \partial_k)$ is the Laplace–Beltrami operator.

A replacement similar to that in (4) is made in the Lagrangian (1) to yield

$$\begin{aligned} L(U_L; \bar{x}, t) = -E\Bigg\{&\frac{f_\pi^2}{4} \mathrm{Tr}[(D_+ U)U^+, (D_- U)U^+] \\ &+ \frac{1}{16e_g^2} \mathrm{Tr}\{[(D_+ U)U^+, (\partial_k U)U^+] \cdot [(D_- U)U^+, (\partial_k U)U^+]\}\Bigg\} \\ &+ \frac{f_\pi^2}{4} \mathrm{Tr}(U_{Lr}, U_{Lr}) + \frac{1}{32e_s^2} \mathrm{Tr}([U_{Lr}, U_{Lb}]^2) \end{aligned} \tag{6}$$

From equation (5) we have

$$\begin{aligned} D_\pm U \equiv D_\pm(A\sigma(x)A^+) = (D_\pm A)\sigma A^+ + A\sigma(D_\pm A^+) \\ \pm 2\nu\, \partial^i A\sigma\, \partial_i A^+ \end{aligned} \tag{7}$$

$$(D_\pm U)U^+ = A[(B_\pm + \sigma B_\pm^+ \sigma^+) \pm 2\nu P^i \sigma P_i \sigma^+]A^+$$

where

$$B_\pm = A^+ D_\pm A(q(t)) = A^+(b^i_\pm\ \partial_i A \pm \nu\Delta_{\mathrm{LB}} A), \qquad P_i = -P_i^+ \equiv A^+\ \partial_i A \tag{8}$$

Taking into account the following relations and definitions introduced following Fujii *et al.* (1986, 1987),

$$
\begin{gathered}
P_i=\frac{i}{2} C_i^B(q)\tau_B, \qquad \hat{\omega}_\pm^B(a)\equiv\hat{b}^i_\pm C_i^B \\
C_i^B C_D^i=\delta_D^B, \qquad C_i^B C_B^k=\delta_i^k \\
\tau_B-\sigma\tau_B\sigma^+=2X_{BD}(\bar{x})\tau_D, \qquad \partial_k\sigma\sigma^+\equiv\frac{i}{2}\xi_k^B(\bar{x})\tau_B \\
\tau_B-\sigma^+\tau_B\sigma=2X_{BD}(\bar{x})\tau_D \\
a(\sigma)\delta_{BE}\equiv\int d^3x\left(f_\pi^2 X_{BD}X_{ED}+\frac{1}{4e_s^2}\xi_k^F\xi_k^H X_{BD}X_{EG}\varepsilon_{FDJ}\varepsilon_{HGJ}\right) \\
g_{ik}=a(\sigma)C_i^B C_k^B
\end{gathered}
\tag{9}
$$

we have

$$
(D_\pm U)U^+=A\left(i\hat{\omega}_\pm^B X_{BD}\tau_D\mp\frac{v}{a}X_{BA}\tau_B\tau_A\right)A^+ \tag{10}
$$

By substitution of (10) into the Lagrangian (6), we obtain

$$
L=\int d^3x\, L(U_L;\bar{x},t)=E[\tfrac{1}{2}g_{ik}(q)\hat{b}^i_+\hat{b}^k_-]-(M_c+\Delta M_c) \tag{11}
$$

where M_c is the standard classical mass of the skyrmion (Fujii *et al.*, 1986, 1987; Adkins *et al.*, 1983) and ΔM_c is a new additional mass term:

$$
\Delta M_c=-\frac{8\pi v^2}{a(\sigma)^2}\int dr\, r^2 s^2\left[f_\pi^2+\frac{1}{2e_s^2}\left(2F'^2+\frac{s^2}{r^2}\right)\right] \tag{12}
$$

with $s=\sin F(r)$, $F'=dF/dr$, $v=1/2$.

The new mass term ΔM_c exactly coincides with that found by Fujii *et al.* (1986, 1987) by an operator quantization method with a modified canonical formalism. Fujii *et al.* (1986, 1987) discussed the role of this term in the stabilization of a rotational skyrmion. In a future paper we will examine the problem of current conservation in a stochastic mechanical approach to skyrmion dynamics.

ACKNOWLEDGMENT

We thank Prof. K. Fujii for useful discussions.

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